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# ANOMALIES FOR ELECTRONS MOVING ON ANTIFERROMAGNETIC SURFACES

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With a complex arrangement of magnetic fields and magnetic couplings, electrons governed by a standard tight-binding Hamiltonian on a square lattice can be made to obey a Dirac equation with an unequal number of positive and negative masses. This induces a spontaneous quantized Hall conductance.

The Dirac equation in one time and two space dimensions exhibits an anomaly [1] in that, in the presence of an electromagnetic field, the vacuum expectation of the current is non-zero. This may be relevant to the explanation of the quantized Hall effect [2]. Another manifestation of this phenomenon is that solitons in antiferromagnetic spin systems, coupled to such a Dirac field, may change statistics and turn into fermions [3]. It is speculated [4] that this may have something to do with high $T_c$ superconductivity. All the above studies involve one or more continuum Dirac fields, such that the number of these fields with positive mass *does not* equal the number of fields with negative mass. For an electric field in the $y$ direction, the induced vacuum current is in the $x$ direction,

$$j_x = \frac{e}{4\pi} \sum_i \operatorname{sign}(m_i) E_y . \quad (1)$$

For this effect to be of interest to problems of condensed matter physics, these electrons must not only obey a Dirac equation, but likewise the number of fields with positive and negative masses must be unequal. It is particularly difficult to achieve the latter due to the persistent doubling on the lattice [5] of fermion species, or of any system whose Hamiltonian is linear in the momenta. Semenoff [6] studied the motion of electrons on a honeycombed graphite lattice and was able to show that, under certain circumstances, each spin state obeys a Dirac equation. The number of positive and negative mass species is the same and the, above mentioned, anomaly does not occur. Recently, Haldane [7] extended the previous model to include an intrinsic, periodic magnetic field, which, when combined with the lattice structure, breaks time reversal invariance; the Dirac fields no longer pair off with opposite sign masses, and a non-zero value for the current of eq. (1) is obtained. This would be a manifestation of a quantized Hall current without the presence of an external magnetic field. As mentioned in ref. [7], the physical realizability of such models is an open question. With the last point in mind we wish to present a different condensed matter situation in which electrons obey a Dirac equation. Unlike the cases in the previous examples, we will work on a simple, square lattice. The Dirac equations will result from an unusual choice of signs of hopping terms, while the mass will appear once we couple the electrons to an antiferromagnetic ordering on the lattice. At this stage we still have an equal number of positive and negative mass Dirac fields. In order to break this symmetry we follow Haldane [7] by introducing a next nearest neighbor hopping term and an internal, periodic magnetic field, with a vanishing net flux.

We shall describe the model in steps. For the moment consider electrons hopping only in the horizontal and vertical directions on the lattice shown in fig. 1; also, for the present ignore the difference between sites marked with a dot or a cross. The sign of an individual hopping term is not physically observable as it may be changed by a site dependent re-

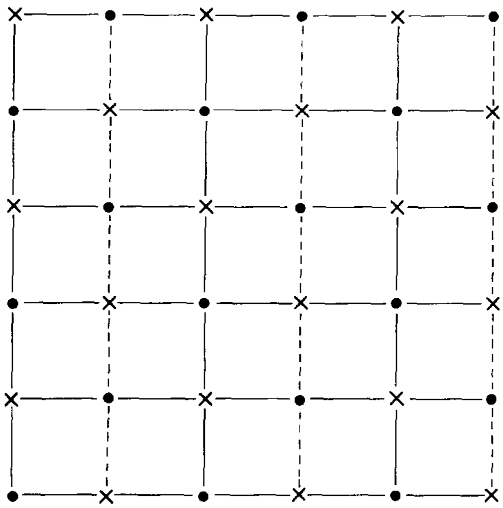

Fig. 1. Nearest neighbor bonds on a square lattice. The sign of the hopping term on the dashed bonds is opposite to that on the solid bonds. • denotes an on site spin up, while × is for spin down.

phasing of the electron fields or gauge transformation. What is gauge invariant is the product of the signs of hopping terms around a unit square. *We shall consider the situation where this product is negative.* This may be achieved by setting the sign of the vertical hopping terms negative along all odd columns and keeping all other hopping terms positive. In fig. 1, the positive hopping terms are denoted by a solid line, the negative ones by a dashed one. Physically, this may be realized weaving a periodic magnetic field through the lattice. If $\Phi$ is the flux through a unit square, the product of the hopping terms around such a square is multiplied by $\exp(ie\Phi)$; for $\Phi=\pm\pi/e$ the usual situation is changed to the one where the product is negative. As the sign of the flux is irrelevant, the above situation can be realized in many ways. Two extreme cases are all signs equal, which could be due to placing the lattice in an external magnetic field, or having the sign of the flux alternate from square to square, in an antiferromagnetic order. *The first case is interesting in that an ordinary tight-binding Hamiltonian turns into a Dirac one in the presence of a constant magnetic field with a half unit of quantized flux through each unit cell.* In the second case the net flux through the lattice is zero.

As we shall see, the above scenario is sufficient to yield a massless Dirac equation. In order to obtain a mass we shall put a magnetic moment on each lattice site, again, fully antiferromagnetically ordered, and couple it to the electrons through a local spin-spin interaction. The dots and crosses on the sites indicate the magnetic moment directions. The Hamiltonian for this model is

$$\begin{aligned} H=&-\frac{t}{2}\sum_{x,y}\chi^{\dagger}_{x,y}[\chi_{x+1,y}+\chi_{x-1,y}\\ &+(-1)^{x}(\chi_{x,y+1}+\chi_{x,y-1})]\\ &+g\sum_{x,y}(-1)^{x+y}\chi^{\dagger}_{x,y}\sigma_z\chi_{x,y}. \end{aligned} \tag{2}$$

In the above, $\chi_{x,y}$ is a two-component electron field at site $x, y$; we take the lattice spacing to be unity and thus $x, y$ are integers. $t$ is the hopping strength and the $(-1)^x$ inside the hopping part of the Hamiltonian is, as discussed above, a result of choosing the product of the signs of the hopping terms to be negative. The strength of the coupling of the electron to the lattice spin is denoted by $g$ and the $(-1)^{x+y}$ is due to the antiferromagnetic ordering.

In terms of a new field, $\psi_{x,y}$, related to $\chi_{x,y}$ by

$$\chi_{x,y}=i^{x+y}(\sigma_y)^x(\sigma_x)^y\psi_{x,y}, \tag{3}$$

the Hamiltonian of eq. (2) takes the form

$$\begin{aligned} H=&-\frac{it}{2}\sum_{x,y}\psi^{\dagger}_{x,y}[\sigma_x(\psi_{x+1,y}-\psi_{x-1,y})\\ &+\sigma_y(\psi_{x,y+1}-\psi_{x,y-1})]\\ &+g\sum_{x,y}\psi^{\dagger}_{x,y}\sigma_z\psi_{x,y}. \end{aligned} \tag{4}$$

We recognize the lattice Dirac Hamiltonian in two space dimensions with a mass proportional to $g$. In momentum space

$$\begin{aligned} H=\int\frac{d^2k}{(2\pi)^2}\,&[t\psi^{\dagger}(k_x,k_y)(\sigma_x\sin k_x\\ &+\sigma_y\sin k_y)\psi(k_x,k_y)+g\psi^{\dagger}(k_x,k_y)\sigma_z\psi(k_x,k_y)]. \end{aligned} \tag{5}$$

The integration runs over the Brillouin zone, $-\pi\leqslant k_{x,y}\leqslant\pi$. The eigenvalues, $\omega$, are given by

$$\omega(k_x,k_y)=\pm\sqrt{g^2+t^2(\sin^2k_x-\sin^2k_y)}. \tag{6}$$

The separation of the two bands is minimal at $(k_x, k_y)=(0, 0)$, $(0, \pi)$, $(\pi, 0)$ and $(\pi, \pi)$. We define four fields:

$$\begin{aligned}
&\psi_1(q_x, q_y)=\psi(q_x, q_y),\\
&\psi_2(q_x, q_y)=\sigma_x\psi(q_x, q_y-\pi),\\
&\psi_3(q_x, q_y)=\sigma_y\psi(q_x-\pi, q_y),\\
&\psi_4(q_x, q_y)=\sigma_z\psi(q_x-\pi, q_y-\pi).
\end{aligned}\tag{7}$$

The continuum Hamiltonian turns into a sum of four Dirac Hamiltonians, two with positive mass and two with negative mass,

$$\begin{aligned}
H=\sum_{i=1}^{4}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\,&[t\psi_i^\dagger(q)\sigma\cdot q\psi_i(q)\\
&+g\eta_i\psi_i^\dagger(q)\sigma_z\psi_i(q)],
\end{aligned}\tag{8}$$

with $\eta_{1,4}=+1$ and $\eta_{2,3}=-1$. At this stage no parity anomalies will occur.

In order to break this symmetry we follow ref. [7] by introducing a next nearest neighbor (nnn) hopping term, $u/4$, and, yet another periodic magnetic field. This one with a vanishing total flux through each unit cell. This configuration is shown in fig. 2; the vector potential, $\boldsymbol{A}$, may be chosen to reside on the nnn links with directions indicated by the arrows; note that with this choice $\nabla\cdot\boldsymbol{A}=0$. With $\phi=e\int\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r}$, and the integral running along the directed nnn links, the term that is added to the Hamiltonian of eq. (2) is

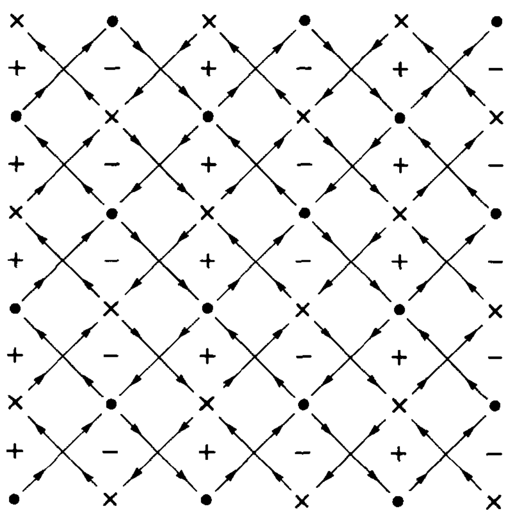

Fig. 2. Next nearest neighbor bonds. (For clarity, the nearest neighbor bonds are not shown.) The arrows show the directions of the vector potential, $\boldsymbol{A}$, and the sign of the flux through various tilted squares is indicated by + and −.

$$\begin{aligned}
H_1=\frac{u}{4}\sum_{x,y}\chi^\dagger_{x,y}[&(\chi_{x+1,y+1}+\chi_{x-1,y-1}\\
&+\chi_{x+1,y-1}+\chi_{x-1,y+1})\cos\phi\\
&+\mathrm{i}(-1)^x(\chi_{x+1,y+1}+\chi_{x-1,y-1}\\
&-\chi_{x+1,y-1}-\chi_{x-1,y+1})\sin\phi].
\end{aligned}\tag{9}$$

Using the related field variables of eq. (3),

$$\begin{aligned}
H_1=\frac{u}{4}\sum_{x,y}\psi^\dagger_{x,y}\sigma_z[&-\mathrm{i}(-1)^x(-\psi_{x+1,y+1}\\
&-\psi_{x-1,y-1}+\psi_{x+1,y-1}+\psi_{x-1,y+1})\cos\phi\\
&-(\psi_{x+1,y+1}+\psi_{x-1,y-1}+\psi_{x+1,y-1}\\
&+\psi_{x-1,y+1})\sin\phi].
\end{aligned}\tag{10}$$

In momentum space, this becomes

$$\begin{aligned}
H_1=-u\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,&\psi^\dagger(k_x, k_y)\sigma_z\\
&\times[\mathrm{i}\psi(k_x+\pi, k_y)\sin k_x\sin k_y\cos\phi\\
&+\psi(k_x, k_y)\cos k_x\cos k_y\sin\phi].
\end{aligned}\tag{11}$$

Using the definition of the four continuum fields in eq. (7) and keeping only terms constant and linear in the momenta, $H_1$ contributes a common mass term to the Hamiltonian of these fields,

$$H_1=-u\sin\phi\sum_{i=1}^{4}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\,\psi_i^\dagger(q)\sigma_z\psi_i(q).\tag{12}$$

Combining this result with the Hamiltonian of eq. (8) we find that fields $\psi_{1,4}$ have masses proportional to $g-u\sin\phi$, while the masses of $\psi_{2,3}$ are proportional to $-g-u\sin\phi$. For $|u\sin\phi|>|g|$ the Hall conductance, $\sigma_{\mathrm{Hall}}=\nu e^2/2\pi\hbar$, with $\nu=\pm2$.

We were able to show that electrons moving on a simple two dimensional lattice with a fairly complex set of antiferromagnetically ordered fields will obey a Dirac equation with an unequal number of positive and negative masses. This will induce a spontaneous quantized Hall conductance. The question remains whether such a situation can be physically realized. In ref. [7] a possible mechanism for inducing such an order was presented.

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